On the Existence Spectrum for Sharply Transitive G-Designs, G a $[k]$-Matching

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Abstract

In this paper we consider decompositions of the complete graph K_v into matchings of uniform cardinality k. They can only exist when k is an admissible value, that is a divisor of $v(v-1)/2$ with $1 \leq k \leq v/2$. The decompositions are required to admit an automorphism group Γ acting sharply transitively on the set of vertices. Here Γ is assumed to be either non-cyclic abelian or dihedral and we obtain necessary conditions for the existence of the decomposition when k is an admissible value with $1 < k < v/2$. Differently from the case where Γ is a cyclic group, these conditions do exclude existence in specific cases. On the other hand we produce several constructions for a wide range of admissible values, in particular for every admissible value of k when v is odd and Γ is an arbitrary group of odd order possessing a subgroup of order $\gcd(k, v)$.

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1 Introduction

For any given graph G, a G-decomposition of the complete graph K_v is a partition $\mathcal{M}$ of the edge-set of K_v into subgraphs, all of which are isomorphic to G. Such a decompositon is also called a G-*design* on v vertices or a G-*design* of order v, see [1]. In the present paper we study G-decompositions of K_v, in which G is the graph given by the vertex disjoint union of k copies of the complete graph K_2. This graph is simply a matching consisting of precisely k edges, briefly a $[k]$-*matching*, and is often denoted by kK_2. Rather than speaking of a kK_2-design, we shall speak of a $[k]$-matching decomposition of K_v or of a $[k]$-matching design on v vertices.

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For a $[k]$-matching on v vertices we have $1 \le k \le v/2$. A $[k]$-matching with $k = 1$ is a single edge and in this case a $[k]$-matching decomposition of K_v is simply the partition of the edge-set of K_v into singletons.

The other extreme case $k = v/2$ occurs precisely when v is even and the $[k]$-matching is a perfect matching or, equivalently, a one-factor. Hence for even v the concept of a $[v/2]$-matching decomposition of K_v coincides with that of a one-factorization of K_v.

A necessary condition for the existence of a $[k]$-matching decomposition of K_v is that k be a divisor of $v(v-1)/2$ with $1 \le k \le v/2$: such a value of k will be called an *admissible* value. It has been proved in [12] that a $[k]$-matching decomposition of K_v exists for each admissible value of k.

We refer to [21] for the standard facts in the theory of finite groups that we shall need. An automorphism group of a $[k]$-matching decomposition $\mathcal{M}$ of K_v is a permutation group Γ on the vertex-set of K_v fixing $\mathcal{M}$ setwise: we shall often say that $\mathcal{M}$ is Γ-invariant meaning that Γ is an automorphism group of $\mathcal{M}$.

We are interested in the situation in which Γ satisfies the additional property of acting *sharply transitively* on the vertices of K_v. That means given arbitrary vertices x and y there exists precisely one automorphism in Γ mapping x to y. An immediate consequence is $|\Gamma| = v$.

In the theory of permutation groups a sharply transitive group is usually called a *regular* permutation group [21, §10.3]: we prefer the terminology *sharply transitive* in our context in order to avoid misunderstandings with the standard use of the term *regular* in graph theory, see also [6], [16].

In their paper [15] Hartman and Rosa define a one-factorization $\mathcal{F}$ of K_v, v even, to be *cyclic* if it admits a cyclic automorphism group Γ acting sharply transitively on vertices. They namely showed that the complete graph K_v, with $v = 2^t$, $t \ge 3$, never admits a cyclic one-factorization; on the other hand, if v is any other even value, they were able to produce a construction for a cyclic one-factorization of K_v.

Bearing the result of Hartman and Rosa in mind, Rees extended the notion of a cyclic one-factorization to that of a cyclic $[k]$-matching decomposition and proved in [19, Thm. 1.2] that the complete graph K_v admits a cyclic $[k]$-matching decomposition for every admissible value of k strictly less than $v/2$.

Any finite group Γ of order v can be realized as a sharply transitive permutation group on v vertices, see [21, Thm. 10.3.1]. This basic elementary observation shows that the existence question for a Γ-invariant $[k]$-matching decomposition of K_v makes perfect sense even when Γ is a non-cyclic group: for even v and $k = v/2$ that is precisely Problem 1.1. in [6]. The same problem can be formulated for arbitrary decompositions of K_v, for instance cycle decompositions, see [7].

Any permutation group Γ on v vertices acts on the edge-set of K_v. If Γ is sharply transitive on vertices, then the Γ-orbit of an edge has either cardinality v or $v/2$, and we speak of a *long* edge and a *long* edge-orbit or of a *short* edge and a *short* edge-orbit, respectively, see [16, §2]. Short edges only exist when v is even, as each such edge is fixed by a uniquely determined involution in the group Γ. A Γ-invariant $[k]$-matching decomposition of K_v exists precisely when the Γ-orbits of edges can be subdivided into subsets in such a way that the union of the orbits in each selected subset can in turn be arranged so as to form a Γ-orbit of $[k]$-matchings: this is essentially the final remark in the Introduction of [15].

For any sharply transitive group Γ on v vertices, the partition of the edge-set of K_v into singletons will obviously form a Γ-invariant $[k]$-matching decomposition of K_v with $k = 1$. If v is even, $k = v/2$ and Γ is non-cyclic abelian, then a Γ-invariant $[v/2]$-matching decomposition of K_v (that is a one-factorization of K_v) exists by [6]. Results on the existence of a one-factorization of K_v, which is sharply transitive with respect to a non-abelian group Γ of even order v, are available only for particular choices of the group Γ. For instance when Γ is dihedral, [3], nilpotent (with additional conditions) [16, 20], a 2-group with an elementary abelian Frattini subgroup, [4], or a symmetric group acting on $2p$ elements for a prime p, [18].

Differently from the even case, when v is odd and $k = (v-1)/2$, a Γ-invariant $\big[(v-1)/2\big]$-matching decomposition of K_v (that is a near one-factorization of K_v) is known to exist for any group Γ of odd order v. This result follows from the existence of the patterned starter in a group Γ of odd order, see for instance [14]: the patterned starter $P(\Gamma)$ is the set $\{\{\gamma, -\gamma\} : \gamma \in \Gamma \setminus \{0_\Gamma\}\}$. The edge $[\gamma, -\gamma]$ is a representative for the edge-orbit $[0, 2\gamma]^\Gamma$ and distinct edges $[\gamma, -\gamma]$ $[\gamma', -\gamma']$ represent distinct edge-orbits.

The assignment of the isomorphism type of the group Γ makes sense also for G-decompositions of graphs other than the complete graph K_v, see for instance [2, 8] when Γ is cyclic or dihedral.

In the present paper we do maintain the choice of a sharply transitive permutation group Γ on v vertices, which is either non-cyclic abelian or dihedral. We shall prove in Sections 3 and 4 that if k is an admissible value with $1 < k < v/2$ satisfying some additional constraint, then a Γ-invariant $[k]$-matching decomposition of K_v does not exist. The additional conditions occur for even v and involve the number of involutions in Γ: they are of purely arithmetic nature and it may well happen that they apply simultaneously to non-isomorphic groups of the same order with equally many involutions, see the final remark of Section 3.

We therefore observe the following phenomenon. If v is a power of 2 and $k = v/2$, then the required $[k]$-matching decomposition does not exist when Γ is cyclic [15], while it does exist when Γ is either non-cyclic abelian [6] or dihedral [3]. Moving from $v/2$ to smaller admissible values of k may lead to the precisely opposite behavior: we namely have existence if Γ is cyclic [19] and, in certain cases, non-existence if Γ is either non-cyclic abelian or dihedral, as we shall see below.

In Section 5 of the paper we construct Γ-invariant $[k]$-matching decompositions for every admissible value of k and for every group Γ of odd order v possessing a subgroup of order $\gcd(k, v)$. This construction can always be performed when Γ is abelian of odd order.

In Section 6 we construct $[k]$-matching decompositions of the required kind under the assumption that Γ is either an elementary abelian 2-group or a dihedral group, with different restrictions on k from case to case. We do have constructions for more groups, for instance those in which the Sylow 2-subgroups satisfy additional properties: we do not insist on illustrating these constructions in detail here. As a matter of fact, we feel that the phenomenon we just described sufficiently shows the different behavior one can expect, depending on the choice of the underlying graph G (say, a perfect or a non-perfect matching, respectively) and of the sharply transitive group Γ (say, a cyclic or a non-cyclic group, respectively).

2 Orbits of $[k]$-matchings

We briefly review some basic facts about sharply transitive decompositions, see [6]. Let $\mathcal{M}$ be a $[k]$-matching decomposition of K_v admitting Γ as an automorphism group acting sharply transitively on vertices. The group Γ will be denoted additively (even if non-abelian) and 0_Γ will denote the zero element of Γ. We can then identify the complete graph K_v with $K_\Gamma = (\Gamma, \binom{\Gamma}{2})$. The action of Γ on the vertices of K_Γ is given by the right regular permutation representation of Γ, that is each element γ of Γ is identified with the permutation $\alpha \mapsto \alpha + \gamma$ of Γ. We shall use the exponential notation for edge-orbits and $[k]$-matching-orbits under the action of Γ and its subgroups.

Let M be a $[k]$-matching in the decomposition $\mathcal{M}$ and denote by Σ the stabilizer of M in Γ. We have that M is the union of the Σ-orbits of its edges and if $[\alpha, \beta]$ is an edge in M, then the stabilizer of $[\alpha, \beta]$ in Γ is contained in Σ; more generally, any element of Γ mapping $[\alpha, \beta]$ to another edge in M is also an element of Σ. We have thus $[\alpha, \beta]^\Sigma = M \cap [\alpha, \beta]^\Gamma$ and the cardinality of $[\alpha, \beta]^\Sigma$ is either $|\Sigma|/2$ or $|\Sigma|$ depending on $[\alpha, \beta]$ being a short or a long edge, respectively.

If $v = |\Gamma|$ is even and M consists of $s > 0$ short edges and $\ell \geq 0$ long edges, the relations $s = s'(|\Sigma|/2)$ and $\ell = \ell'|\Sigma|$ hold for some integers $s' > 0$, $\ell' \geq 0$, whence $k = s + \ell = s'(|\Sigma|/2) + \ell'|\Sigma| = (|\Sigma|/2)(s' + 2\ell')$ and so, in particular, $(|\Sigma|/2)$ is a divisor of k.

If $v = |\Gamma|$ is odd or if v is even and M contains no short edges, then $s = 0$ and the relation $k = \ell'|\Sigma|$ holds for some positive integer ℓ'.

We emphasize the following immediate consequence.

Proposition 1. *If v is even and $[\alpha, \beta]$ is a short edge, then the cardinality of the Σ-orbit $[\alpha, \beta]^\Sigma$ is a divisor of* $\gcd(k, v/2)$. *If v is odd or if v is even* *and M only contains long edges, then the cardinality of the Σ-orbit $[\alpha,\beta]^{\Sigma}$ is a divisor of* $\gcd(k,v)$.

For the rest of this section we assume v even.

Proposition 2. *Any two conjugate involutions in Γ fix equally many edges in any given edge-orbit.*

Proof. The assertion follows immediately from the observation that the involution τ fixes the edge $[\alpha,\beta]$ if and only if the conjugate involution $-\gamma+\tau+\gamma$ fixes the edge $[\alpha+\gamma,\beta+\gamma]$. □

Proposition 3. *Let τ_1 and τ_2 be involutions of Γ from distinct conjugacy classes. If τ_1 fixes the edge $[\alpha,\beta]$ then τ_2 fixes no edge in $[\alpha,\beta]^{\Gamma}$.*

Proof. Assume there exists an edge in $[\alpha,\beta]^{\Gamma}$ which is fixed by τ_2, that is there exists $\gamma\in\Gamma$ such that $([\alpha,\beta]+\gamma)+\tau_2=[\alpha,\beta]+\gamma$. That means $\gamma+\tau_2-\gamma$ fixes the edge $[\alpha,\beta]$. But the unique involution fixing the short edge $[\alpha,\beta]$ is $\tau_1=-\alpha+\beta$, whence $\gamma+\tau_2-\gamma=\tau_1$, a contradiction since τ_2 and τ_1 are not conjugate in Γ. □

Proposition 4. *Let $[\alpha,\beta]$ be a short edge and let M be a $[k]$-matching in $\mathcal{M}$ having a non-empty intersection with $[\alpha,\beta]^{\Gamma}$. If the cardinality $|M\cap[\alpha,\beta]^{\Gamma}|$ is odd then M has an empty intersection with each short edge-orbit whose edges are fixed by involutions which are not conjugate to $-\alpha+\beta$ in Γ. That is the case, in particular, if $d=\gcd(k,v/2)$ is odd.*

Proof. As remarked before Proposition 1, we have $k=(|\Sigma|/2)(s'+2\ell')$ and $|\Sigma|/2$ is the cardinality of each Σ-orbit in M consisting of short edges, hence $|M\cap[\alpha,\beta]^{\Gamma}|=|\Sigma|/2$ is a divisor of both k and $|\Gamma|/2=v/2$, whence also a divisor of d.

Assume M contains a short edge which is fixed by an involution, say τ, which is not conjugate to $-\alpha+\beta$ in Γ. By Proposition 3 we have that τ fixes no edge in $[\alpha,\beta]^{\Gamma}$, whence exchanges the edges in $[\alpha,\beta]^{\Gamma}$ in pairs. In particular, since τ fixes M, we have that τ exchanges in pairs the edges in $M\cap[\alpha,\beta]^{\Gamma}$. Then $|M\cap[\alpha,\beta]^{\Gamma}|$ is even, a contradiction. □

As an immediate consequence of Propositions 2 and 4, the following statement holds.

Proposition 5. *Let $\tau\in\Gamma$ be an involution fixing some edge in the short edge-orbit $\mathcal{E}$ and let Υ be the conjugacy class of τ in Γ. Let M be a $[k]$-matching in $\mathcal{M}$ with some edge in $\mathcal{E}$. If $d=\gcd(k,v/2)$ is odd, then M has a non-empty intersection with at most $|\Upsilon|$ short edge-orbits containing edges which are fixed by involutions in Υ.*

It is not surprising that the behavior of involutions plays a role in the existence problem for sharply transitive $[k]$-matching decompositions. Involutions are responsible for short edge-orbits, which need to fit well with each other and with long edge-orbits. Indeed similar phenomena occur in the existence problems for 1-rotational 2-factorizations of complete graphs, see [9], and for 2-pyramidal 2-factorizations of the cocktail party graph, see [10].

Let $\mathcal{M}$ be a Γ-invariant $[k]$-matching decomposition of K_v. The group Γ partitions the elements of $\mathcal{M}$ into Γ-orbits and in each such orbit we can select a representative, called a *base block*, and obtain a complete system of distinct representatives for the Γ-orbits of $\mathcal{M}$. Knowledge of the base blocks is sufficient for a complete description of the Γ-invariant $[k]$-matching decomposition.

The notion of a partial difference family is a useful tool for the construction of a complete system of base blocks. We recall the definition of a partial difference family in the case of $[k]$-matchings (we refer to [5] for a more general definition).

Let M be a $[k]$-matching of K_Γ. If Σ is the stabilizer of M in Γ, then M is the union of the Σ-orbits of its edges, namely $M = [\alpha_1, \beta_1]^\Sigma \cup \ldots \cup [\alpha_t, \beta_t]^\Sigma$, $1 \leq t \leq k$. We set $Z = \{[\alpha_i, \beta_i] : 1 \leq i \leq t\}$ and denote by X the subset of Z given by the short edges $[\alpha_i, \beta_i] \in Z$ which are fixed by the involution $-\alpha_i + \beta_i$ belonging to Σ.

We define the list of partial differences of M as the multiset

$$\partial M = \left(\cup_{[\alpha_i,\beta_i]\in X} \{\alpha_i - \beta_i\}\right) \cup \left(\cup_{[\alpha_i,\beta_i]\in Z \setminus X} \{\alpha_i - \beta_i, \beta_i - \alpha_i\}\right).$$

Given a set $\mathcal{B}$ of $[k]$-matchings of K_Γ, we define the multiset $\partial\mathcal{B} = \cup_{M\in\mathcal{B}} \partial M$.

Following [5], we say that a set $\mathcal{B}$ of $[k]$-matchings of K_Γ is a (Γ, kK_2)-partial difference family (PDF for short) if $\partial\mathcal{B} = \Gamma \setminus \{0_\Gamma\}$ meaning that this is an equality of sets (and so, in particular $\partial\mathcal{B}$ has no repeated elements). By [5], the following result holds.

Proposition 6. *The existence of a (Γ, kK_2)-PDF is equivalent to the existence of a $[k]$-matching decomposition of the complete graph K_v admitting Γ as an automorphism group acting sharply transitively on vertices.*

The elements of a (Γ, kK_2)-PDF are the base blocks of the decomposition.

3 Non-cyclic abelian groups of even order: non-existence

In this section Γ will denote an abelian group of even order v in its sharply transitive permutation representation. We assume Γ has precisely m involutions, hence m is also the number of distinct conjugacy classes of involutions in Γ. In this situation Proposition 5 can be formulated as follows.

Proposition 7. *Let $\mathcal{M}$ be a Γ-invariant $[k]$-matching decomposition of K_v. If $d = \gcd(k, v/2)$ is odd, then each $[k]$-matching M in $\mathcal{M}$ has a non-empty intersection with at most one short edge-orbit.* $\square$

Proposition 7 yields the following necessary condition.

Proposition 8. *Let $\mathcal{M}$ be a Γ-invariant $[k]$-matching decomposition of K_v. If $d = \gcd(k, v/2)$ is odd, then for the number m of involutions in Γ we have $mk \leq d(v-1)$.*

Proof. Let $\{M_1, \ldots, M_r\}$ be a complete system of distinct representatives for the Γ-orbits of $[k]$-matchings in $\mathcal{M}$. For each $i = 1, \ldots, r$, let Σ_i denote the stabilizer of M_i in Γ.

From the fact that $\mathcal{M}$ is a Γ-invariant partition of the edge-set, it follows that for each short edge-orbit there is a unique $[k]$-matching among $M_1, \ldots,$ M_r intersecting it non-trivially. From the remark above and from Proposition 7 we have that precisely m of the $[k]$-matchings $M_1, \ldots, M_r$, say $M_{e_1}, \ldots, M_{e_m}$, have a non-empty intersection with some short edge-orbit. Observe that $1 \leq m \leq r$ and we can relabel indices so that $M_{e_i} = M_i$ holds for every $i = 1, \ldots, m$.

For the $[k]$-matching-orbit M_i^Γ we have $|M_i^\Gamma| = |\Gamma : \Sigma_i| = v/|\Sigma_i|$. Now $|\Sigma_i|$ is even as Σ_i contains involutions, so we can write this relation in the form $|M_i^\Gamma| = (v/2)/(|\Sigma_i|/2)$, showing that $|\Sigma_i|/2$ is a divisor of $v/2$. The discussion before Proposition 1 shows that $|\Sigma_i|/2$ is also a divisor of k. We can therefore write $|M_i^\Gamma| = v/(2d_i)$ for some divisor d_i of d. We have $v/(2d) \leq v/(2d_i)$ for every $i = 1, \ldots, m$, and the relation $|\mathcal{M}| \geq |M_1^\Gamma \cup \ldots \cup M_m^\Gamma|$ yields $\frac{v(v-1)}{2k} \geq \frac{v}{2d_1} + \ldots + \frac{v}{2d_m} \geq m\frac{v}{2d}$, whence our assertion. $\square$

In case d is even, which necessarily implies $v \equiv 0 \pmod 4$, a $[k]$-matching M in $\mathcal{M}$ might contain short edges which are fixed by involutions from distinct conjugacy classes in Γ. This circumstance yields a somewhat weaker inequality as the next statement shows.

Proposition 9. *Assume $v \equiv 0 \pmod 4$ and let $\mathcal{M}$ be a Γ-invariant $[k]$-matching decomposition of K_v. If $d = \gcd(k, v/2)$ is even, then for the number m of involutions in Γ we have $mk < 2d^2(v-1)$.*

Proof. Let $\{M_1, \ldots, M_r\}$ be a complete system of distinct representatives for the Γ-orbits of $[k]$-matchings in $\mathcal{M}$. For each $i = 1, \ldots, r$, let Σ_i denote the stabilizer of M_i in Γ and let s_i' denote the number of short Σ_i-orbits in M_i.

Since $\mathcal{M}$ is a Γ-invariant partition of the edge-set, given a short edge-orbit there is a unique $[k]$-matching among $M_1, \ldots, M_r$ intersecting it non-trivially. Denote by $M_{e_1}, \ldots, M_{e_\mu}$ those $[k]$-matchings among $M_1, \ldots, M_r$

having a non-empty intersection with some short edge-orbit. Observe that $1 \leq \mu \leq m \leq r$ and we can relabel indices so that $M_{e_i} = M_i$ holds for every $i = 1, \dots, \mu$, whence also $s'_1 + \ldots + s'_\mu = m$.

Fix an arbitrary index i with $1 \leq i \leq \mu$. For each one of the s'_i short edge-orbits intersecting M_i, the corresponding involution fixing the edges in the orbit lies in Σ_i, whence $s'_i < |\Sigma_i|$.

As in the proof of Proposition 8, we have $|M_i^\Gamma| = (v/2)/(|\Sigma_i|/2)$, showing that $|\Sigma_i|/2$ is a divisor of $v/2$. The discussion before Proposition 1 shows that $|\Sigma_i|/2$ is also a divisor of k. We can thus write $|\Sigma_i| = 2d_i$ for some divisor d_i of d, yielding $s'_i < |\Sigma_i| = 2d_i \leq 2d$, whence also $m = s'_1 + \ldots + s'_\mu < \mu 2d$, or, equivalently, $\mu > m/(2d)$.

We have $|M_i^\Gamma| = v/(2d_i)$ for every $i = 1, \dots, m$, and so the relation $|\mathcal{M}| \geq |M_1^\Gamma \cup \cdots \cup M_\mu^\Gamma|$ yields

$$\frac{v(v-1)}{2k} \geq \frac{v}{2d_1} + \ldots + \frac{v}{2d_\mu} \geq \mu \frac{v}{2d} > \frac{m}{2d}\frac{v}{2d},$$

whence our assertion. □

We exhibit instances in which the bounds of Propositions 8, 9 rule out the existence of a Γ-invariant $[k]$-matching decomposition of K_v.

We know from the structure theorem for finite abelian groups, see for instance [21, Ch.5], that if Γ is a finite abelian group of even order v then all of its involutions lie in the unique Sylow 2-subgroup Θ, which, in turn, is the direct sum of cyclic groups of 2-power order. If w is the number of such direct summands, then the number m of involutions is given by $m = 2^w - 1$. We have $m = 1$ if and only if $w = 1$, that is if and only if there is just one direct summand, that is if and only if Θ is cyclic. If that is the case, the inequalities of Propositions 8 and 9 become $k \leq d(v-1)$ and $k < 2d^2(v-1)$, respectively, and are weaker than $k \leq v/2$. These inequalities may therefore become meaningful only for $m > 1$, that is when the Sylow 2-subgroup of Γ is non-cyclic.

In the next statement we apply the bounds of Propositions 8 and 9 to the special situation in which Γ is an elementary abelian 2-group.

Proposition 10. *Let Γ be an elementary abelian group of order $v = 2^n \geq 16$. Let k be an admissible value, with $1 < k < \frac{v}{2}$, written in the form $k = 2^r c$ with c odd. If either $r = 0$ or r is positive with $c > 2^{r+1}$, then there exists no Γ-invariant $[k]$-matching decomposition of K_v.*

Proof. Assume there exists a Γ-invariant $[k]$-matching decomposition of K_v. In the former case, since $k = c$ is odd and $v = 2^n$, we have $d = \gcd(k, v/2) = 1$, hence Proposition 8 yields $(v-1)k \leq (v-1)$, a contradiction. In the latter case we have $d = \gcd(k, v/2) = 2^r$ and Proposition 9 yields $(v-1)k < 2 \cdot 2^{2r}(v-1)$, whence $c < 2^{r+1}$, a contradiction. □

For $v = 2$ or 4 there are no admissible values of k with $1 < k < v/2$; $k = 2$ is the only admissible value with $1 < k < v/2$ for $v = 8$ and a [2]-matching decomposition of K_8 which is invariant under the elementary abelian group of order 8 does exist, as we shall see in Section 6.

Table 1 lists some situations for which existence is excluded by the previous results. An entry of the form $n_1^{r_1} \dots n_s^{r_s}$ in the column labelled with Γ means the isomorphism type of the group is

$$\underbrace{\mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_1}}_{r_1} \quad \oplus \quad \underbrace{\mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_2}}_{r_2} \quad \oplus \quad \underbrace{\ldots}_{\ldots} \quad \oplus \quad \underbrace{\mathbb{Z}_{n_s} \oplus \ldots \oplus \mathbb{Z}_{n_s}}_{r_s}$$

v	Γ	m	k
16	$2^2\,4$	7	3
16	2^4	15	$3, 5$
40	$2^3\,5$	7	13
56	$2^3\,7$	7	11
64	2^6	63	$3, 7, 9, 14, 18, 21$
88	$2^3\,11$	7	29
96	$2^2\,24, 4^2\,6$	7	19
96	$2^3\,12$	15	19
96	$2^5\,3$	31	$5, 15, 19$
112	$2^2\,28, 2 \cdot 4 \cdot 14$	7	37
112	$2^4\,7$	15	37
256	2^8	255	$3, 5, 10, 15, 17, 30, 34, 51, 60, 68, 85, 102$
320	$2^3\,40, 2^2 \cdot 4 \cdot 20$	15	$29, 145$
320	$2^4\,20$	31	$11, 29, 145$
320	$2^6\,5$	63	$11, 29, 55, 58, 145$
512	2^9	511	$7, 14, 73, 146$
1024	2^{10}	1023	$3, 11, 22, 31, 33, 44, 62, 66, 93,$ $124, 132, 186, 248, 264, 341, 372$

Table 1: Some non-existence values for abelian groups

The criteria introduced in Propositions 8 and 9 are purely numerical. Non-isomorphic abelian groups of the same order with the same number of involutions will therefore yield the same set of admissible values to which these criteria apply simultaneously: each multiple entry in the second column of Table 1 is an instance of such a situation.

4 Dihedral groups: non-existence

Let $\varGamma$ denote the dihedral group of order v, with v even, $v \geq 6$. The group $\varGamma$ is uniquely determined up to isomorphims and, in additive notation, we may define $\varGamma$ as the group generated by elements ρ and ε of order $v/2$ and 2, respectively, subject to the relation $-\varepsilon + \rho + \varepsilon = -\rho$. If $v/2$ is odd, then $\varGamma$ possesses a unique conjugacy class of involutions, namely $\varUpsilon = \{i\rho + \varepsilon : i = 1, \ldots, v/2\}$. If $v/2$ is even, the conjugacy classes of involutions in $\varGamma$ are $\varUpsilon_0 = \{(v/4)\rho\}$, $\varUpsilon_1 = \{2i\rho + \varepsilon : i = 1, \ldots, v/4\}$ and $\varUpsilon_2 = \{(2i + 1)\rho + \varepsilon : i = 0, \ldots, v/4 - 1\}$.

Let $\mathcal{M}$ be a $\varGamma$-invariant $[k]$-matching decomposition of K_v. We shall say for short that $\mathcal{M}$ is a *dihedral* $[k]$-matching decomposition of K_v and define $d = \gcd(k, v/2)$

Let $\{M_1, \ldots, M_r\}$ be a complete system of distinct representatives for the $\varGamma$-orbits of $[k]$-matchings in $\mathcal{M}$. For each $i = 1, \ldots, r$, let $\varSigma_i$ denote the stabilizer of M_i in $\varGamma$, let s'_i denote the number of short $\varSigma_i$-orbits in M_i and let ℓ'_i denote the number of long $\varSigma_i$-orbits in M_i.

Since $\mathcal{M}$ is a $\varGamma$-invariant partition of the edge-set, given a short edge-orbit there is a unique $[k]$-matching among $M_1, \ldots, M_r$ intersecting it non-trivially. Denote by $M_{e_1}, \ldots, M_{e_\mu}$ those $[k]$-matchings among $M_1, \ldots, M_r$ having a non-empty intersection with some short edge-orbit. We can relabel indices so that $M_{e_i} = M_i$ holds for every $i = 1, \ldots, \mu$.

Fix an arbitrary index i with $1 \leq i \leq \mu$. For each one of the s'_i short edge-orbits intersecting M_i, the corresponding involution fixing the edges in the orbit lies in $\varSigma_i$, whence $s'_i < |\varSigma_i|$. As for the abelian case (see for instance the proof of Proposition 8) we can write $|\varSigma_i| = 2d_i$ for some divisor d_i of d yielding $s'_i < |\varSigma_i| = 2d_i < 2d$.

We have $|M_i^\varGamma| = v/(2d_i) \geq v/(2d)$ for every $i = 1, \ldots, \mu$ and so the relation $|\mathcal{M}| \geq |M_1^\varGamma \cup \cdots \cup M_\mu^\varGamma|$ yields $\frac{v(v-1)}{2k} \geq \mu \frac{v}{2d}$, whence $\mu \leq \frac{d(v-1)}{k}$.

Assume d odd. By Proposition 5 a $[k]$-matching in $\{M_1, \ldots, M_\mu\}$ can only contain short edges which are fixed by conjugate involutions in $\varGamma$. If $v/2$ is even, i.e. $\varGamma$ has three conjugacy classes of involutions, then among $M_1, \ldots, M_\mu$ there is exactly one $[k]$-matching, say M', intersecting only the short edge-orbit of $[0, (v/4)\rho]$, while there are $t \geq 1$ such $[k]$-matchings, say $M_{f_1}, \ldots, M_{f_t}$, containing only edges which are fixed by involutions in $\varUpsilon_1$ and $\mu-(t+1) \geq 1$ such $[k]$-matchings containing only short edges which are fixed by involutions in $\varUpsilon_2$. We can relabel indices so that $M' = M_1$, $M_{f_i} = M_{i+1}$ hold for $i = 1, \ldots, t$ and the remaining $[k]$-matchings are $M_{t+2}, \ldots, M_\mu$. We have thus $s'_1 = |\varUpsilon_0| = 1$, $v/2 = |\varUpsilon_1| + |\varUpsilon_2| = s'_2 + \cdots + s'_\mu < 2d(\mu-1)$, whence also $v/(4d) + 1 < \mu \leq (d(v-1))/k$. If $v/2$ is odd, i.e. $\varGamma$ possesses a unique conjugacy class of involutions, then we have $v/2 = |\varUpsilon| = s'_1 + \ldots + s'_\mu < 2d\mu$, whence also $v/(4d) < \mu \leq (d(v-1))/k$.

Assume d even. We have $v \equiv 0 \pmod 4$ and $\varGamma$ possesses three conjugacy classes of involutions. A $[k]$-matching in $\mathcal{M}$ might contain short edges which are fixed by non-conjugate involutions in $\mathit{\Gamma}$ (see Proposition 5). We get thus no benefit from sorting the $[k]$-matchings $M_1, \ldots, M_\mu$ according to the conjugacy classes Υ_0, Υ_1, Υ_2 and obtain only $v/2+1 = s'_1 + \cdots + s'_\mu < 2d\mu$, whence $(v+2)/(4d) < \mu \leq d(v-1)/k$.

Proposition 11. *If k is an admissible value and $d = \gcd(k, v/2)$ is such that $k \geq 4d^2$ holds, then there exists no dihedral $[k]$-matching decomposition of K_v.*

Proof. Suppose there exists a dihedral $[k]$-matching decomposition $\mathcal{M}$ of K_v and consider the inequalities arising from the previous discussion.

If $v \equiv 0 \pmod 4$ and d is odd, then the inequalities $v/(4d) + 1 < \mu \leq d(v-1)/k$ yield $k < 4d^2(v-1)/(v+4d) < 4d^2$, a contradiction.

If $v \equiv 0 \pmod 4$ and d is even, then the inequalities $(v+2)/(4d) < \mu \leq d(v-1)/k$ yield $k < 4d^2(v-1)/(v+2) < 4d^2$, a contradiction.

Finally, in case $v \equiv 2 \pmod 4$ holds, then the inequalities $v/(4d) < \mu \leq d(v-1)/k$ yield $k < 4d^2(v-1)/v < 4d^2$, a contradiction. □

Table 2 lists the values of k for which existence is excluded by the previous proposition for $v \leq 100$.

5 Groups of odd order: some constructions

As mentioned in Section 1, the patterned starter of a group $\mathit{\Gamma}$ of odd order plays a role in the costruction of a $\mathit{\Gamma}$-invariant near one-factorization of the complete graph K_v. It can be employed also in the construction of a one-factorization of K_v which is 1-rotational with respect to a group of odd order $v-1$. We recall that a one-factorization $\mathcal{F}$ of K_v is 1-rotational with respect to a group $\mathit{\Gamma}$ if $\mathit{\Gamma}$ leaves $\mathcal{F}$ invariant, fixes one vertex of K_v and acts sharply transitively on the remaining ones. The well-known construction of Lucas, [17], provides an example of a one-factorization which is 1-rotational with respect to the cyclic group of order $v-1$. It can be generalized to algebraic systems other than cyclic groups, see [11]. This generalization becomes straightforward for (abelian or non-abelian) groups of odd order, because we can use patterned starters.

In the proof of Proposition 12 we use patterned starters to construct $\mathit{\Gamma}$-invariant $[k]$-matching decompositions of the complete graph when $\mathit{\Gamma}$ is a group of odd order possessing a subgroup of order $\gcd(k, v)$. The construction can always be performed when $\mathit{\Gamma}$ is abelian. This fact shows that the situation for abelian groups of odd order is radically different from the even order case, in which non-existence may occur as we have seen in Section 3.

The use of patterned starters in the construction of sharply transitive near one-factorizations or 1-rotational one-factorizations does not involve any particular arithmetic or algebraic constraint, as it does happen in the

v	k	$4d^2$
16	5	4
22	7	4
26	5	4
28	9	4
34	11	4
36	$5, 7$	4
40	13	4
46	$5, 9, 15$	4
50	7	4
52	17	4
56	$5, 11, 22$	$4, 16$
58	19	4
64	$7, 9, 18$	$4, 16$
66	$5, 13$	4
70	23	4
76	$5, 15, 25, 30$	$4, 16$
78	$7, 11$	4
82	$9, 27$	4
86	$5, 17$	4
88	29	4
94	31	4
96	$5, 19, 38$	$4, 16$
100	$9, 11, 18, 22, 33$	$4, 16$

Table 2: Some non-existence values for the dihedral group

construction of $[k]$-matching decompositions (see the proof of Proposition 12).

We recall some basic facts about patterned starters in view of the proof of Proposition 12. As already outlined in Section 1, the patterned starter of an additive group Γ of odd order is the set $P(\Gamma)$ of all possible unordered pairs $\{\gamma, -\gamma\}$ with $\gamma \in \Gamma \smallsetminus \{0_\Gamma\}$. Each pair $\{\gamma, -\gamma\}$ is thus an edge of the complete graph K_Γ and we can also write $P(\Gamma) = \{[\gamma, -\gamma] : \gamma \in \Gamma \smallsetminus \{0_\Gamma\}\}$. The edge $[\gamma, -\gamma]$ is a representative for the edge-orbit $[0, 2\gamma]^\Gamma$; two distinct edges represent distinct edge-orbits (this property follows from the fact that every element of Γ has odd order). Therefore, the pairs of a patterned starter provide a complete system of distinct representatives for the Γ-orbits of edges. Furthermore, the following holds.

Lemma 1. *Let Γ be a group of odd order v and let Σ be a subgroup of Γ. The patterned starter $P(\Gamma)$ contains a subset S of $\big((v/|\Sigma|)-1\big)/2$ pairs $\{\gamma, -\gamma\}$, with $\gamma \notin \Sigma$, providing a complete system S' of distinct representatives for the non-trivial left cosets of Σ in Γ. The set S' can be written in the form $Z \cup (-Z)$ where $Z = \{\gamma : \{\gamma, -\gamma\} \in S\}$, $-Z = \{-\gamma : \gamma \in Z\}$.*

Proof. Let γ be an element of $\Gamma \smallsetminus \Sigma$, we show that γ and $-\gamma$ belong to distinct left cosets of Σ in Γ. Suppose $\gamma + \Sigma = -\gamma + \Sigma$, then $2\gamma \in \Sigma$; since $\langle 2\gamma \rangle$ is a subgroup of Σ and $\langle 2\gamma \rangle = \langle \gamma \rangle$, then γ is in Σ, a contradiction. Therefore γ and $-\gamma$ represent distinct left cosets of Σ in Γ and consequently the set of non-trivial left cosets of Σ in Γ can be written in the form $X \cup (-X)$, where $-X = \{-\gamma + \Sigma : \gamma + \Sigma \in X\}$. It is straightforward to see that the set $S = \{\{\gamma, -\gamma\} : \gamma + \Sigma \in X\}$ is a subset of $P(\Gamma)$ of cardinality $\big((v/|\Sigma|) - 1\big)/2$, yielding a complete system of distinct representatives for the non-trivial left cosets of Σ in Γ, say S'. S' can be written in form $Z \cup (-Z)$, where $Z = \{\gamma : \{\gamma, -\gamma\} \in S\}$ and $-Z = \{-\gamma : \gamma \in Z\}$. $\square$

In the next statement we construct a sharply transitive $[k]$-matching decomposition of K_v with respect to an arbitrary group Γ of order v possessing a subgroup of order $h = \gcd(v, k)$.

Proposition 12. *Let Γ be a group of odd order v and let k be an admissible value. If Γ has a subgroup Σ of order $h = \gcd(v, k)$, then there exists a $[k]$-matching decomposition of K_v admitting Γ as an automorphism group acting sharply transitively on vertices.*

Proof. Let $P(\Gamma) = \{\{\gamma, -\gamma\} : \gamma \in \Gamma \smallsetminus \{0_\Gamma\}\}$ be the patterned starter of Γ and let Σ be a subgroup of Γ of order h. We set $v/h = 2z + 1$. By Lemma 1, $P(\Gamma)$ contains a subset S of z pairs $\{\gamma, -\gamma\}$, with $\gamma \notin \Sigma$, providing a complete system S' of distinct representatives for the non-trival left cosets of Σ in Γ. S' can be written in the form $Z \cup (-Z)$, with $|Z| = |-Z| = z$.

We set $f = \gcd(k, (v-1)/2)$, whence $k = fh$ since v and $(v-1)/2$ are coprime. We also set $(v-1)/(2f) = hq + r$ where q and r are non-negative integers with $r < h$. The relations $fh = k \leq (v-1)/2$ yield $f \leq (v-1)/(2h) < v/(2h)$, that is $f \leq z$ since f and z are positive integers. Therefore we can take a subset Φ of Z with $|\Phi| = f$; since $r < h$, we can also take a subset Ω of Σ with $|\Omega| = r$.

For every $\omega \in \Omega$ we set $A_\omega = \{[\phi+\omega, -(\phi+\omega)] : \phi \in \Phi\}$ and denote by A'_ω the corresponding subset of $P(\Gamma)$, namely $A'_\omega = \{\{\phi+\omega, -(\phi+\omega)\} : \phi \in \Phi\}$. The set A_ω is an $[f]$-matching of K_v, since the elements in $\Phi \cup (-\Phi)$ are distinct representatives for the left cosets of Σ in Γ. For the same reason, $A'_\omega \cap A'_{\omega'} = \emptyset$ for every pair of distinct elements $\omega, \omega' \in \Omega$, therefore the set $A' = \cup_{\omega\in\Omega} A'_\omega$ is a subset of $P(\Gamma)$ of cardinality fr.

For every $\omega \in \Omega$, we denote by M_ω the Σ-orbit of A_ω, namely $M_\omega = A_\omega^\Sigma = \cup_{\phi\in\Phi}[\phi+\omega, -(\phi+\omega)]^\Sigma$. Since the vertices of A_ω represent distinct left cosets of Σ in Γ, each Σ-orbit $[\phi+\omega, -(\phi+\omega)]^\Sigma$ is an $[h]$-matching and M_ω can be viewed as the vertex-disjoint union of f $[h]$-matchings, that is M_ω is a $[k]$-matching.

Consider the set $Q = P(\Gamma) \smallsetminus A'$. Since Q has cardinality $\big((v-1)/2\big) - fr = kq + fr - fr = kq$, we can partition Q into q sets of cardinality k, say $N'_1, \ldots, N'_q$, each of which yields a $[k]$-matching of K_v, say $N_1, \ldots, N_q$, respectively.

We show that the set $\mathcal{B} = \{M_\omega : \omega \in \Omega\} \cup \{N_1, \ldots, N_q\}$ is a (Γ, kK_2)-PDF, that is we show that $\partial\mathcal{B}$ is the set of non-zero elements of Γ. We shall use the property that distinct pairs in $P(\Gamma)$ represent distinct Γ-orbits of edges, that is $\{\pm(2\gamma)\} \cap \{\pm(2\gamma)'\} = \emptyset$ for every $\{\gamma, -\gamma\}$, $\{\gamma', -\gamma'\} \in P(\Gamma)$, $\gamma \neq \pm\gamma'$. Note that: (i) $\partial M_\omega = \partial A_\omega$, as Σ is the stabilizer of M_ω in Γ, whereas A_ω has a trivial stabilizer in Γ; (ii) each list of partial differences does not contain repeated elements.

Since the elements in $A' \cup \{N'_1, \ldots, N'_q\}$ are pairwise disjoint, the corresponding elements in $\mathcal{B}$ have disjoint lists of partial differences. Furthermore, since $A' \cup \{N'_1, \ldots, N'_q\}$ partitions $P(\Gamma)$ and the pairs in $P(\Gamma)$ provide a complete system of distinct representatives for the Γ-orbits of edges, the set $\partial\mathcal{B}$ is the set of non-zero elements of Γ. It is thus proved that $\mathcal{B}$ is a (Γ, kK_2)-PDF. The assertion follows from Proposition 6. $\square$

As the converse of Lagrange's Theorem is true for finite abelian groups, [21], we see that each finite abelian group Γ of order v does possess a subgroup of order $h = \gcd(k, v)$, hence Proposition 12 holds for Γ and can be written as follows.

Proposition 13. *Let Γ be an abelian group of odd order v and let k be an admissible value. There exists a $[k]$-matching decomposition of K_v admitting Γ as an automorphism group acting sharply transitively on vertices.* $\square$

We give an example showing how to construct a Γ-invariant $[k]$-matching decomposition of K_v using Proposition 12.

We consider $v = 21$; the admissible values for k are: 2, 3, 5, 7 and 10. For $k = 10$, that is $k = (v-1)/2$, there always exists a [10]-matching decomposition of K_v with respect to an arbitrary group of order $v = 21$ (see the remarks in Section 1 about near one-factorizations). An abelian group of order 21 is necessarily cyclic and the result of Rees [19, Thm. 1.2] guarantees the existence of a cyclic $[k]$-matching decomposition of K_{21} for $k \in \{2, 3, 5, 7\}$. There is a unique non-abelian group Γ of order 21, which can be generated by two elements γ_1 and γ_2 of order 3 and 7, respectively, subject to the single relation (in additive notation) $-\gamma_1 + \gamma_2 + \gamma_1 = 2\gamma_2$, see for instance the `AllSmallGroup` library of GAP, [13]. We construct a Γ-invariant $[k]$-matching decomposition of K_{21} for every $k \in \{2, 3, 5, 7\}$. The elements of Γ can be written in the form $i\gamma_1 + j\gamma_2$, with $0 \le i \le 2$, $0 \le j \le 6$ and the patterned starter of Γ is the set:

$$P(\Gamma) = \{\{\gamma_1, 2\gamma_1\}, \{\gamma_2, 6\gamma_2\}, \{2\gamma_2, 5\gamma_2\}, \{3\gamma_2, 4\gamma_2\}, \{\gamma_1+\gamma_2, 2\gamma_1+3\gamma_2\}, \\ \{\gamma_1+2\gamma_2, 2\gamma_1+6\gamma_2\}, \{\gamma_1+3\gamma_2, 2\gamma_1+2\gamma_2\}, \{\gamma_1+4\gamma_2, 2\gamma_1+5\gamma_2\}, \\ \{\gamma_1+5\gamma_2, 2\gamma_1+\gamma_2\}, \{\gamma_1+6\gamma_2, 2\gamma_1+4\gamma_2\}\}.$$

Following the proof of Proposition 12, we see that in order to obtain the base blocks of a Γ-invariant [2]-matching decomposition of K_{21} it suffices to partition $P(\Gamma)$ into 2-subsets. We can construct the following base blocks:

$$\begin{aligned}
N_1 &= \{[\gamma_1, 2\gamma_1], [\gamma_2, 6\gamma_2]\}\\
N_2 &= \{[2\gamma_2, 5\gamma_2], [3\gamma_2, 4\gamma_2]\}\\
N_3 &= \{[\gamma_1+\gamma_2, 2\gamma_1+3\gamma_2], [\gamma_1+2\gamma_2, 2\gamma_1+6\gamma_2]\}\\
N_4 &= \{[\gamma_1+3\gamma_2, 2\gamma_1+2\gamma_2], [\gamma_1+4\gamma_2, 2\gamma_1+5\gamma_2]\}\\
N_5 &= \{[\gamma_1+5\gamma_2, 2\gamma_1+\gamma_2], [\gamma_1+6\gamma_2, 2\gamma_1+4\gamma_2]\}
\end{aligned}$$

For $k = 3$, the group Γ possesses a subgroup of order $\gcd(3, 21) = 3$, namely $\Sigma = \langle\gamma_1\rangle$. Following the proof of Proposition 12, we can set $Z = \{\gamma_2, 2\gamma_2, 3\gamma_2\}$, $\Phi = \{\gamma_2\}$, $\Omega = \{\gamma_1\}$ and obtain the following base blocks:

$$\begin{aligned}
M_{\gamma_1} &= [\gamma_2+\gamma_1, -(\gamma_2+\gamma_1)]^{\langle\gamma_1\rangle} = \{[\gamma_1+2\gamma_2, 2\gamma_1+6\gamma_2], [2\gamma_1+4\gamma_2, 5\gamma_2], [\gamma_2, \gamma_1+3\gamma_2]\}\\
N_1 &= \{[\gamma_1, 2\gamma_1], [\gamma_2, 6\gamma_2], [2\gamma_2, 5\gamma_2]\}\\
N_2 &= \{[3\gamma_2, 4\gamma_2], [\gamma_1+\gamma_2, 2\gamma_1+3\gamma_2], [\gamma_1+3\gamma_2, 2\gamma_1+2\gamma_2]\}\\
N_3 &= \{[\gamma_1+4\gamma_2, 2\gamma_1+5\gamma_2], [\gamma_1+5\gamma_2, 2\gamma_1+\gamma_2], [\gamma_1+6\gamma_2, 2\gamma_1+4\gamma_2]\}
\end{aligned}$$

For $k = 5$ we partition $P(\Gamma)$ into 5-subsets and obtain the following base blocks:

$$\begin{aligned}
N_1 = \ & \{[\gamma_1, 2\gamma_1], [\gamma_2, 6\gamma_2], [2\gamma_2, 5\gamma_2], [3\gamma_2, 4\gamma_2], [\gamma_1+\gamma_2, 2\gamma_1+3\gamma_2]\}\\
N_2 = \ & \{[\gamma_1+2\gamma_2, 2\gamma_1+6\gamma_2], [\gamma_1+3\gamma_2, 2\gamma_1+2\gamma_2], [\gamma_1+4\gamma_2, 2\gamma_1+5\gamma_2],\\
& [\gamma_1+5\gamma_2, 2\gamma_1+\gamma_2], [\gamma_1+6\gamma_2, 2\gamma_1+4\gamma_2]\}
\end{aligned}$$

For $k = 7$ the group Γ possesses a subgroup of order $\gcd(7, 21) = 7$, namely $\Sigma = \langle\gamma_2\rangle$. Following the proof of Proposition 12 we can set $Z = \Phi = \{\gamma_1\}$, $\Omega = \{0_\Gamma, \gamma_2, 2\gamma_2\}$ and obtain the following set of base blocks:

$M_{0_\Gamma} = [\gamma_1, 2\gamma_1]^{\langle\gamma_2\rangle} = \{[\gamma_1, 2\gamma_1], [\gamma_1+\gamma_2, 2\gamma_1+\gamma_2], [\gamma_1+2\gamma_2, 2\gamma_1+2\gamma_2],$
$[\gamma_1+3\gamma_2, 2\gamma_1+3\gamma_2], [\gamma_1+4\gamma_2, 2\gamma_1+4\gamma_2], [\gamma_1+5\gamma_2, 2\gamma_1+5\gamma_2], [\gamma_1+6\gamma_2, 2\gamma_1+6\gamma_2]\}$;
$M_{\gamma_2} = [\gamma_1+\gamma_2, 2\gamma_1+3\gamma_2]^{\langle\gamma_2\rangle} = \{[\gamma_1+\gamma_2, 2\gamma_1+3\gamma_2], [\gamma_1+2\gamma_2, 2\gamma_1+4\gamma_2],$
$[\gamma_1+3\gamma_2, 2\gamma_1+5\gamma_2], [\gamma_1+4\gamma_2, 2\gamma_1+6\gamma_2], [\gamma_1+5\gamma_2, 2\gamma_1],$
$[\gamma_1+6\gamma_2, 2\gamma_1+\gamma_2], [\gamma_1, 2\gamma_1+2\gamma_2]\}$;
$M_{2\gamma_2} = [\gamma_1+2\gamma_2, 2\gamma_1+6\gamma_2]^{\langle\gamma_2\rangle} = \{[\gamma_1+2\gamma_2, 2\gamma_1+6\gamma_2], [\gamma_1+3\gamma_2, 2\gamma_1],$
$[\gamma_1+4\gamma_2, 2\gamma_1+\gamma_2], [\gamma_1+5\gamma_2, 2\gamma_1+2\gamma_2], [\gamma_1+6\gamma_2, 2\gamma_1+3\gamma_2],$
$[\gamma_1, 2\gamma_1+4\gamma_2], [\gamma_1+\gamma_2, 2\gamma_1+5\gamma_2]\}$;
$N_1 = \{[\gamma_2, 6\gamma_2], [2\gamma_2, 5\gamma_2], [3\gamma_2, 4\gamma_2], [\gamma_1+3\gamma_2, 2\gamma_1+2\gamma_2], [\gamma_1+4\gamma_2, 2\gamma_2+5\gamma_2],$
$[\gamma_1+5\gamma_2, 2\gamma_2+\gamma_2], [\gamma_1+6\gamma_2, 2\gamma_2+4\gamma_2]\}$.

6 Groups of even order: some constructions

Propositions 10 and 11 show non-existence for certain admissible values when Γ is an elementary abelian group or a dihedral group, respectively. In the present section we prove that, while maintaining Γ elementary abelian or dihedral, a different choice of admissible values does yield existence.

If $v = 2^n$ and Γ is an elementary abelian group of order v in its sharply transitive permutation representation, then a Γ-invariant $[k]$-matching decomposition of K_v is also called *elementary abelian.*

Proposition 14. *Assume $v = 2^n$, with $n > 1$. If k is an admissible value dividing v, then there exists an elementary abelian $[k]$-matching decomposition of K_v.*

Proof. Let k be a divisor of 2^{n-1} and let Γ be the elementary abelian group of order $v = 2^n$. For every non-zero element $\gamma \in \Gamma$, there exists a subgroup of Γ of order k not containing γ, say Σ_γ. It is straightforward to see that the set $\mathcal{B} = \{[0_\Gamma, \gamma]^{\Sigma_\gamma} : \gamma \in \Gamma \setminus \{0_\Gamma\}\}$ is a (Γ, kK_2)-PDF. The assertion follows from Proposition 6. □

Let Γ denote the dihedral group of order v, v even and $v \geq 6$. As remarked in Section 4, the group Γ can be considered in additive notation as the group generated by the elements ρ and ε of order $v/2$ and 2, respectively, subject to the relation $-\varepsilon + \rho + \varepsilon = -\rho$. The following statement holds.

Proposition 15. *Let $v \geq 6$ be an even integer and let k be a divisor of $v/2$ with $1 < k < v/2$. Then there exists a dihedral $[k]$-matching decomposition of K_v.*

Proof. Let $\Gamma = \langle\rho, \varepsilon\rangle$ be the dihedral group of order v. By the main result in [19], there exists a $\langle\rho\rangle$-invariant $[k]$-matching decomposition $\mathcal{M}$ of $K_{v/2}$. Let $\{M_1, \ldots, M_t\}$ be a set of base blocks for $\mathcal{M}$ and let $\Pi = \langle v/(2k)\rho\rangle$

be the subgroup of $\langle\rho\rangle$ of order k. It is straightforward to see that the set $\mathcal{B} = \{M_1, \ldots, M_t\} \cup \{[0_\Gamma, \gamma]^\Pi : \gamma \in \Gamma \setminus \langle\rho\rangle\}$ is a (Γ, kK_2)-PDF. The assertion follows from Proposition 6. $\square$

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